Smart point landmark distribution for thin-plate splines

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ABSTRACT

Landmark placement is crucial in manual demarcation and registration of anatomical structures, registration of different image modalities (i.e. MRI/CT), labeling training data for lip and face principal component models, training for neural networks, and signal interpolation to name some applications. Although landmark placement at curvature and coordinate extrema (e.g. corners of the mouth, lowest point on the lower lip) is fairly unambiguous, the placement of point landmarks along a linear contour is subjective. Unfortunately the user's choice of landmark placement determines the quality of the resulting registration. In this paper, we present an algorithm to remove these undesired degrees of freedom by re-placing landmarks along the contour. Ambiguous landmarks are moved so as to minimize a thin plate spline energy while constraining the landmarks to the originally specified contour. The resulting landmark placement results in a smoother registration while still interpolating the contours and fixed landmarks. The results show that the ambiguity of manual landmark placement along contours does affect the smoothness of the interpolated registration, and that significantly smoother interpolations can be achieved using our approach. This procedure may also benefit other applications employing landmarks by eliminating unintended curvature (variation) from the landmark data.

Keywords: Landmark placement, registration, morphing, thin plate splines, radial basis functions

1. INTRODUCTION

Point landmarks are used in a wide variety of registration and interpolation applications. Interpolation algorithms generally assume that the landmark positions are known exactly, but in real applications the localization of landmarks is always prone to some error. Errors in placing point features have been addressed by regularized interpolation (Section 2), which provides a smoother function that approximates rather than strictly interpolating the data.[1] While this approach handles placement problems for point features, many applications require the placement of point landmarks along contour features.

Although landmark placement at curvature and coordinate extrema (e.g. corners of the mouth, lowest point on the lower lip) is fairly unambiguous, the placement of point landmarks along a contour is quite arbitrary (Figure 1). Unfortunately this arbitrary choice of landmark placement determines the quality of the resulting registration. This issue has been addressed with algorithms that base registration on contour rather than point data.[2] In such schemes the issue of point placement along a contour is merely replaced by the issue of how to parameterize the contours, however. Obvious parameterization schemes such as arc length parameterization will not result in smooth or appropriate results in some cases. Also, contour drawing requires more sophisticated user skills and editing tools than does point editing.

In this paper, these undesired degrees of freedom are minimized by re-placing landmarks along a contour. The algorithm produces the least-curvature spline consistent with the intended contours and fixed points. We distinguish "slidable" (contour-constrained) landmarks from unambiguous (fixed) landmarks (Figure 1). The latter may include corners (curvature extrema), extreme points (coordinate extrema), and symmetry points that are unambiguous in visual inspection. Contour-constrained landmarks are those that require a subjective judgment regarding where along the contour they are placed.

We adjust the placement of the contour-constrained landmarks along their respective contours so as to produce the smoothest thin plate spline (TPS) that interpolates the contours and fixed landmarks. While this removes an undesired

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ambiguity in the case of TPS registration, there are other applications requiring manually placed landmarks (such as labeling training data for active appearance models) where thin-plate splines are not used. In some such applications, we can *define* the desired landmark placement to be the one that produces the smoothest thin-plate registration, thus introducing the TPS specifically for the purpose of smart landmark placement, and thereby eliminating subjectively introduced variation from the landmark data.

2. BACKGROUND

Thin plate splines (TPS) interpolate specified points while minimizing an approximate curvature (integrated squared second derivative),

$$\int \left(\frac{\partial^2 f}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 f}{\partial y^2}\right)^2 dxdy \tag{1}$$

resulting in a smooth deformation without unexpected ripples and variations. Thin plate splines have been used in morphometrics research to define changes of shape between subjects of the same species,[3] and in neurosurgery and radiotherapy research for the purpose of defining a non-rigid registration from patient specific radiographs to generic atlas representations.[4]

When the number of points to be interpolated is small (as is typically the case with manually placed landmarks) there is a simple radial basis function (RBF) formulation.

The radial basis formulation of TPS is

$$\delta(\mathbf{p}) = \sum_k^n c_k |\mathbf{p} - \mathbf{p}_k|^2 \log |\mathbf{p} - \mathbf{p}_k| + a_1 + (a_2, a_3)\mathbf{p} \tag{2}$$

where $\delta(\mathbf{p})$ is the desired displacement at a point $\mathbf{p}$, $\mathbf{p}_k$ are the n landmarks (each with a corresponding destination landmark $\mathbf{t}_k$), and a_j are coefficients of an affine registration (the measure (1) is invariant to the offset and linear slope of the optimized function so separate terms are added to handle this).

The weights $c_k, a_j, k = 1 \ldots n, j = 1 \ldots 3$ needed to interpolate the data can be found by solving the block matrix system

$$\begin{bmatrix} \mathcal{K} & \mathcal{P} \\ \mathcal{P}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \delta \\ 0 \end{bmatrix} \tag{3}$$

where $\mathcal{K}_{r,c} = ||p_r - p_c||^2 \log(p_r - p_c),\ r, c \in 1 \ldots n$, $\mathcal{P}$ is a n by 3 matrix containing the constant (1) vector and the landmark locations x_k and y_k, and δ are the data values to be interpolated. In a registration or 2D labeling application there are two such interpolations needed, one for each of x, y, but the matrix $\mathcal{K}$ depends only on the data locations (not their values) so the matrix inverse is needed only once.

In practice it is desirable to "regularize" the system (3). Regularization can be motivated both in terms of matrix conditioning and weight decay considerations, and by assuming that the landmarks have some error associated with them. In practice it involves adding an amount λI (= 0.001 for example) to the diagonal of $\mathcal{K}$ before inverting.[1] This has the effect of reducing large coefficients, thereby producing a more planar warp.

3. METHOD

Our objective is to minimize the squared (approximate) curvature (integral of second derivative squared) of the spline, the same criterion that TPS minimizes. In a radial basis warp a single *local* RBF function adds a constant amount of approximate curvature. Scaling this function scales the total curvature. So a squared coefficient c^2 adds an amount c^2 of curvature.

So if the basis function were local, we could minimize the TPS criterion by minimizing the sum square of the RBF coefficients c_k. The TPS basis functions $r^2 log(r)$ in (2) are not local, but most of their curvature is concentrated near the origin, so we can use the sum-square coefficients as an approximation (in particular, if the coefficients c_k are all zero, the resulting surface is planar). The coefficient energy can be taken as $(\sum c_k)^2$ or as $\sum c_k^2 = \mathbf{c}^T \mathbf{c}$; we choose the latter for simplicity.

In summary, we wish to minimize $\mathbf{c}^T\mathbf{c}$ by adjusting the slidable points, while constraining them to the contour defined by the originally specified points. The coefficient energy can be minimized by moving landmarks in the range of the "warp", the domain, or both. We choose the former, and derive a constrained gradient descent adjustment of the contour-constrained landmarks.

3.1. Details

Say that p is a particular landmark in the source image, t_1 is the corresponding destination landmark, v is a unit vector through t_1 toward the adjacent landmark on one side, and t is an undetermined point on this line. The objective of minimizing the coefficient energy while being constrained to the line v can be expressed as

$$\min_t \mathbf{c}^T\mathbf{c} + \lambda((t - t_1) \cdot n) \tag{4}$$

where n is the unit perpendicular vector (normal) to v and λ is a Lagrange multiplier enforcing the constraint. In the sequel we wish to work with the entire contour rather than altering a single landmark at a time. The objective is similar,

$$\min_{\mathbf{t}} \mathbf{c}^T\mathbf{c} + \lambda^T N(\mathbf{t} - \mathbf{t}_1) \tag{5}$$

with $\mathbf{t}, \mathbf{t}_1, \mathbf{p}$ now being vectors containing all landmark points, with the x- and y-components in some consistent order, such as all the x's followed by all the y's (bold variables now denote vectors that reference all the landmarks), and N is a $n \times 2n$ matrix where the $2k, 2k+1$ elements of each row contains the x, y components of the normal n for the corresponding line in $\mathbf{t} - \mathbf{t}_1$.

Next $\mathbf{c}$ needs to be expressed in terms of an altered landmark location $\mathbf{t}$.

$$\delta = \begin{bmatrix} \mathbf{t}_x - \mathbf{p}_x \\ \mathbf{t}_y - \mathbf{p}_y \end{bmatrix} = \begin{bmatrix} \mathcal{K} & 0 \\ 0 & \mathcal{K} \end{bmatrix} \begin{bmatrix} \mathbf{c}_x \\ \mathbf{c}_y \end{bmatrix} + \begin{bmatrix} \mathcal{P} & 0 \\ 0 & \mathcal{P} \end{bmatrix} \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \end{bmatrix}$$

or more compactly,

$$\mathbf{t} - \mathbf{p} = \mathbf{Kc} + \mathbf{Pa}$$

where the vector on the left contains the x displacements to interpolate followed by the y displacements (dimension $2n$), $\mathbf{K}$ is of size $(2n)^2$, $\mathbf{c}$ is of size $2n$ containing the x coefficients of the radial basis TPS followed by the y coefficients, $\mathbf{P}$ is of size $2n$ by 6, and $\mathbf{a}$ is of size 6 and contains the three affine coefficients for the x-component of the warp followed by the three y coefficients (note that $\mathbf{K}$ and $\mathbf{P}$ now refer to the block diagonal matrices that contain the original $\mathcal{K}, \mathcal{P}$). Continuing,

$$\begin{aligned} &\mathbf{t} - \mathbf{p} - \mathbf{Pa} = \mathbf{Kc} \\ &\mathbf{c} = (\mathbf{K}^T\mathbf{K})^{-1}\mathbf{K}^T(\mathbf{t} - \mathbf{p} - \mathbf{Pa}) \\ &\ = \mathbf{M}(\mathbf{t} - \mathbf{p} - \mathbf{Pa}) \ \text{ with } \mathbf{M} \equiv (\mathbf{K}^T\mathbf{K})^{-1}\mathbf{K}^T = \mathbf{K}^{-1} \end{aligned}$$

This expression for $\mathbf{c}$ can be substituted into the objective (5)

$$\min_{\mathbf{t}} (\mathbf{t} - \mathbf{p} - \mathbf{Pa})^T\mathbf{M}^T\mathbf{M}(\mathbf{t} - \mathbf{p} - \mathbf{Pa}) + \lambda^T N(\mathbf{t} - \mathbf{t}_1)$$

Taking the derivative we get

$$2\mathbf{M}^T\mathbf{M}\mathbf{t} - 2\mathbf{M}^T\mathbf{M}\mathbf{p} - 2\mathbf{M}^T\mathbf{M}\mathbf{P}\mathbf{a} + N^T\lambda \tag{6}$$

as the gradient with respect to $\mathbf{t}$.

3.2. Lambda

To find λ, (6) can be solved for $\mathbf{t}$ noting that the gradient is zero at the minimum, and this can be substituted into

$$N(\mathbf{t} - \mathbf{t}_1) = 0 \tag{7}$$

The function of λ is simply to keep the new landmarks $\mathbf{t}$ on the original line segments, however. This can also be achieved simply by moving $\mathbf{t}$ down its gradient (excluding the constraint) and then projecting it back on the contour.

3.3. Algorithm

With the gradient now defined, the landmark re-placement algorithm iteratively moves the contour-constrained landmarks a small distance down the gradient until the gradient approaches zero. In this procedure the landmarks are sliding along the piecewise linear contour defined by the original (unadjusted) landmarks. At each iteration, it is necessary to check each moving landmark to see if it has moved beyond the end of one of the segments of the original contour — if so, it should be moved back to the adjacent original landmark.

4. RESULTS

The input landmark locations for a sample registration application are shown in Figure 2. Figure 3 shows the landmarks on the radiograph as moved by the coefficient-minimizing optimization. The x- and y- components of a registration can be considered individually as height field functions x=f(u,v), y=g(u,v). In Figure 4 we plot the x- and y- height fields defined by TPS using the original and adjusted landmarks. The functions interpolating the adjusted landmarks are visibly smoother. Figure 5 shows a regular grid warped by the TPS, before (left) and after (right) landmark adjustment. The grid on the right is visibly smoother in several regions.

5. CONCLUSIONS

TPS interpolants are often selected specifically because they produce the smoothest possible interpolation, but our results show that the ambiguity of manually placing landmarks along contours can interfere with this goal. The algorithm described in this paper adjusts such contour-constrained landmarks so as to remove subjectively introduced variation. While this procedure will directly benefit TPS registration applications, it may also benefit other applications employing landmarks by eliminating unintended curvature (variation) from the landmark data.

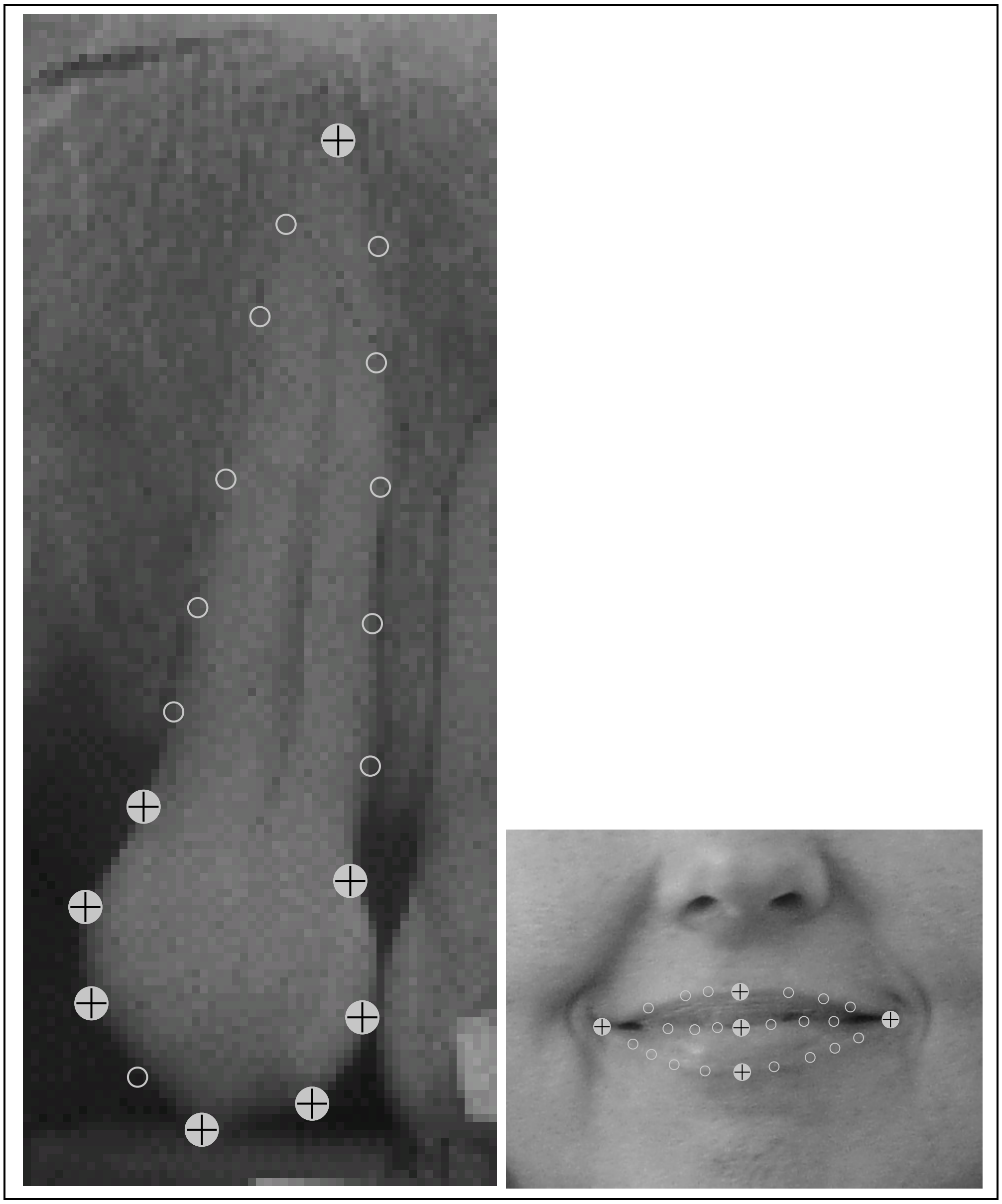

Figure 1. While some "features" such as curvature extrema, coordinate extrema, and midpoints can be objectively located (filled circles), placing landmarks along contours is intrinsically ambiguous (open circles). We refer to the open circles as "slidable" or contour-constrained landmarks.

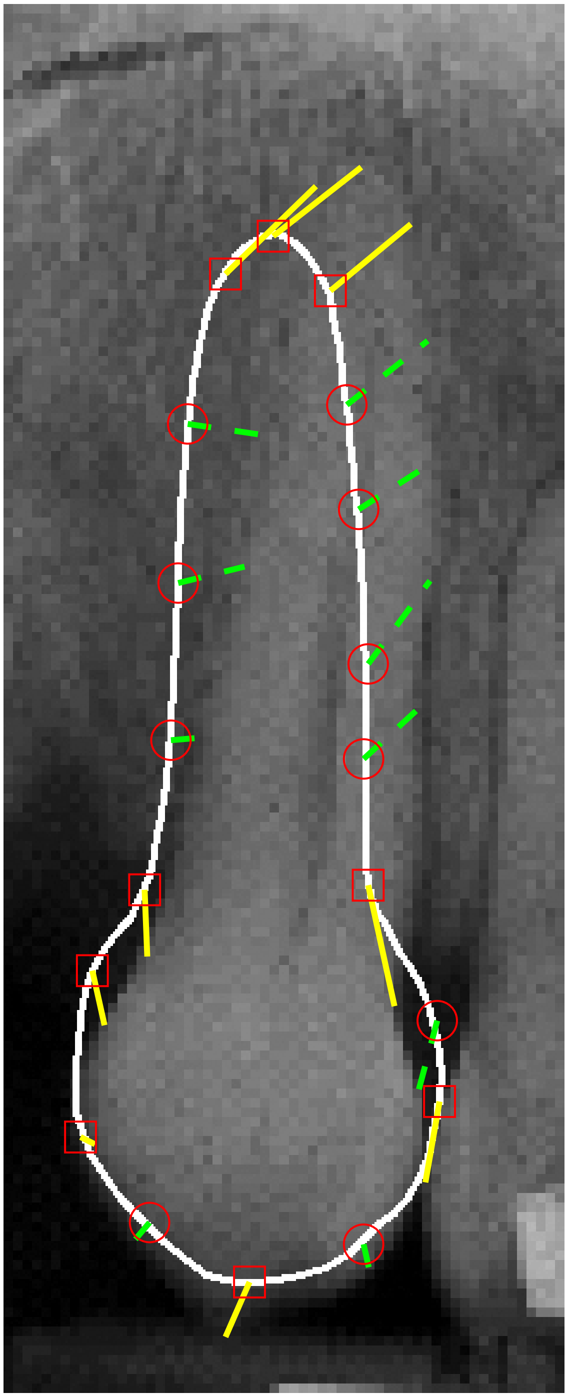

Figure 2. Registration of a generic tooth prototype (white outline) to a particular tooth (radiograph image). Squares and solid lines indicate corresponding unambiguous landmarks; circles and dashed lines indicate contour-constrained landmarks.

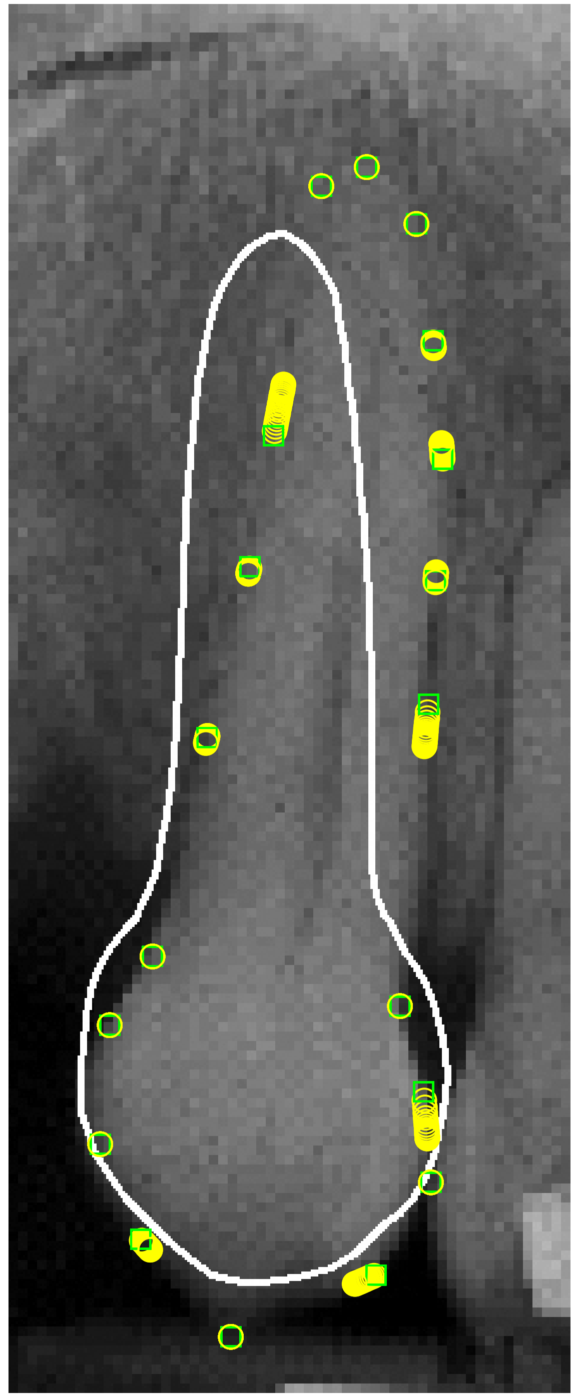

Figure 3. Target landmarks are moved (circles) from their original position (squares) to produce the smoothest possible warp, constrained to the original contour. The combined x,y RBF coefficient "energy" was reduced from 46.12 to 12.30 over 20 iterations.

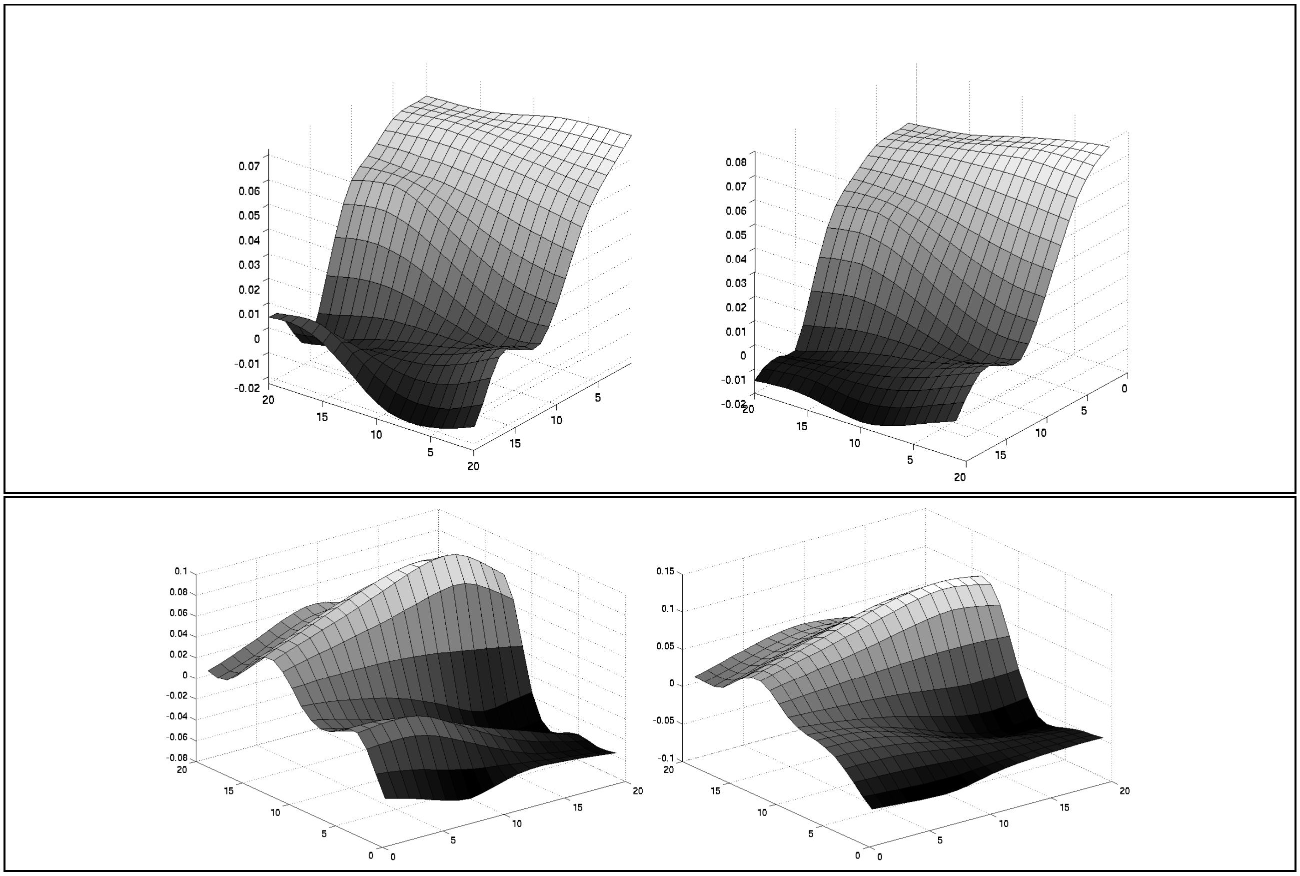

Figure 4. Height field functions computed by TPS interpolation of the original (left) and adjusted (right) landmarks. The functions at the top are the x-component of the morph; the bottom pair are the y-component. The interpolation of the adjusted landmarks is visibly smoother, yet the same points and contours are interpolated

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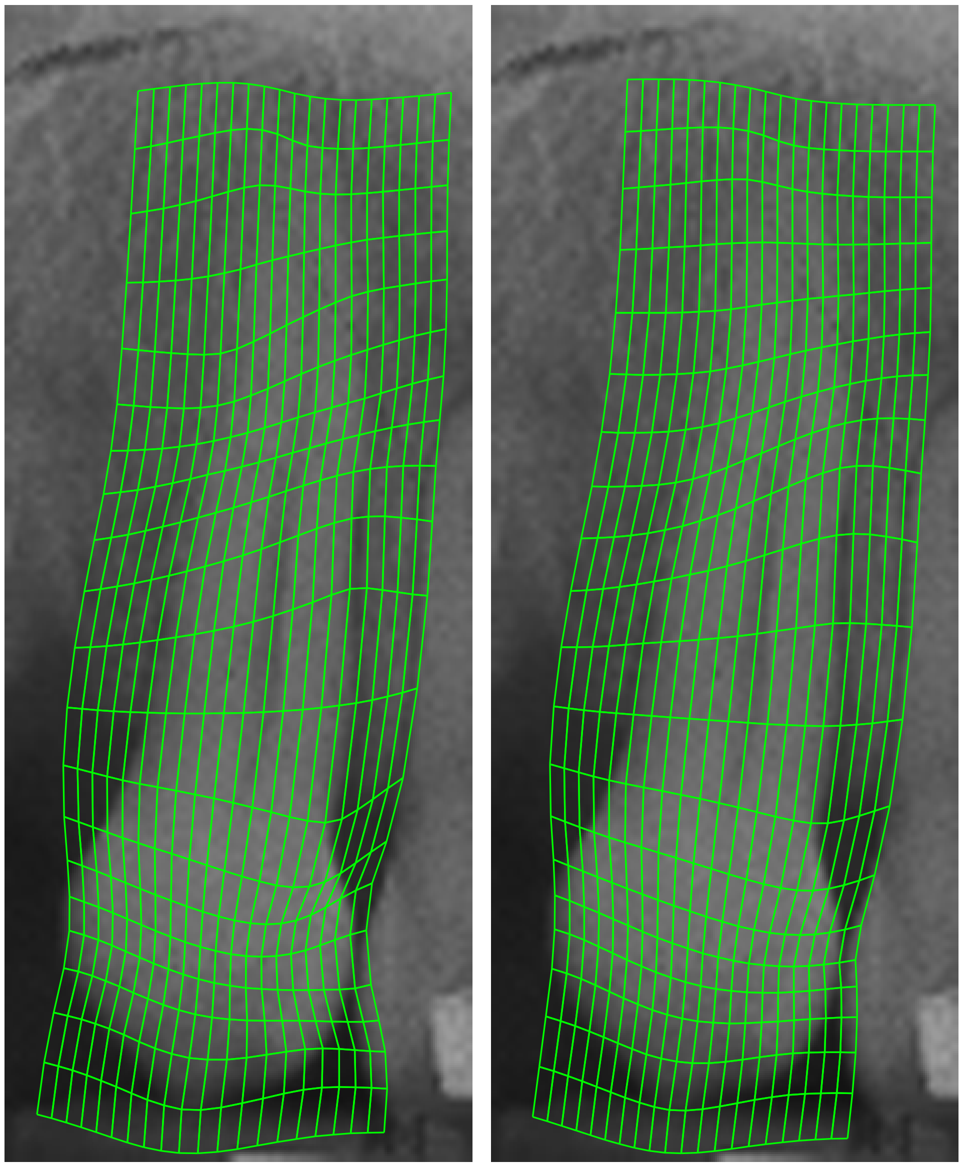

Figure 5. Computed thin plate warp applied to a regular grid. Left (original), right (after contour landmark movement). The warp on the right is noticeably smoother in some areas.